On a Conjecture of Andrica & Tomescu

Blair D. Sullivan[1]
Oak Ridge National Laboratory
1 Bethel Valley Road, Oak Ridge, TN 37831
sullivanb@ornl.gov

Abstract

Let $S(n)$ be the integer sequence which is the coefficient of $x^{n(n+1)/4}$ in the expansion of $(1+x)(1+x^2)\cdots(1+x^n)$ for positive integers n congruent to 0 or 3 mod 4. We prove a conjecture of Andrica and Tomescu [1] that $S(n)$ is asymptotic to $\sqrt{6/\pi}\cdot 2^n n^{-3/2}$ as n approaches infinity.

1 Introduction

Let $S(n)$ denote the coefficient of the middle term of the expansion of the polynomial $(1+x)(1+x^2)\cdots(1+x^n)$ when $n \equiv 0$ or 3 mod 4 (otherwise $n(n+1)/4$ is not an integer, and the expansion has no middle term). Andrica and Tomescu conjectured that as n approaches infinity, $S(n)$ behaves asymptotically like $\sqrt{6/\pi}\cdot 2^n n^{-3/2}$. More formally, writing $f(n) \sim g(n)$ to denote

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1,$$

we have

Conjecture 1. *[Andrica, Tomescu [1]] $S(n) \sim \sqrt{6/\pi}\cdot 2^n n^{-3/2}$ for $n \equiv 0$ or 3 mod 4.*

From [1], one can write $S(n)$ in integral form via Cauchy's formula as

$$S(n) = \frac{2^{n-1}}{\pi}\int_0^{2\pi} \cos(t)\cos(2t)\cdots\cos(nt)\,dt.$$

We will use the Laplace method to estimate this integral [2]. Rewriting, we have $S(n) = \frac{2^{n-1}}{\pi}\int_0^{2\pi} f_n(t)\,dt$ where $f_n(t) = \prod_{k=1}^{n}\cos(kt)$. In Section 2, we analyze the behavior of $f_n(t)$ and note a technical lemma needed for the main proof of Conjecture 1, which is presented in Section 3.

[1] The submitted manuscript has been authored by a contractor of the U.S. Government under Contract No. DE-AC05-00OR22725. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.

2 Behavior of $f_n(t)$

Lemma 2. *Let $0 < \varepsilon < 1/4$, and $f_n(t) = \prod_{k=1}^{n} \cos(kt)$. Then $\int_{n^{-(3/2-\varepsilon)}<|t|<\pi/2} |f_n(t)|\, dt = o(n^{-3/2})$ as $n \to \infty$.*

Proof. We break the integral into three pieces based on the value of $|t|$.

Case 1. $n^{-(\frac{3}{2}-\varepsilon)} \le |t| \le \frac{1}{n}$*:*

Since $\cos(x) = \cos(-x)$, and cos is a monotone decreasing function on $[0, \pi]$, $f_n(t) = f_n(-t)$ is also monotone decreasing for $t \in [0, 1/n]$, and it suffices to give an appropriate upper bound on $f_n(n^{-(3/2-\varepsilon)})$.

Since we need $\int_{n^{-(3/2-\varepsilon)}<|t|<\pi/2} f_n(t)\, dt = o(n^{-3/2})$, given that $0 < \varepsilon < 1/4$, it suffices to show that for a constant $c > 0$,

$$f_n(n^{-(3/2-\varepsilon)}) \le \exp(-cn^{2\varepsilon}(1 + o(1))).$$

Using the Taylor series expansion, we know $\cos(kt) \le 1 - \frac{(kt)^2}{2!} + \frac{(kt)^4}{4!}$. Substitution then yields

$$f_n(t) = \prod_{k=1}^{n} \cos(kt) \le \prod_{k=1}^{n} \left(1 - \frac{(kt)^2}{2!} + \frac{(kt)^4}{4!}\right),$$

since $k \le n$ and $|t| \le 1/n$ implies $kt \le 1$. When $t = n^{-(3/2-\varepsilon)}$, we have

$$f_n(t) \le \prod_{k=1}^{n} \left(1 - \frac{k^2 n^{-(3-2\varepsilon)}}{2} + \frac{k^4 n^{-(6-4\varepsilon)}}{24}\right).$$

To evaluate, we note the terms of this product are all in $[0, 1]$, and apply $\log(1 - x) \le -x$:

$$\log \prod_{k=1}^{n} \left(1 - \left(\frac{k^2 n^{-(3-2\varepsilon)}}{2} - \frac{k^4 n^{-(6-4\varepsilon)}}{24}\right)\right) \le \sum_{k=1}^{n} \left(-\frac{k^2 n^{-(3-2\varepsilon)}}{2} + \frac{k^4 n^{-(6-4\varepsilon)}}{24}\right).$$

Writing $\sum_{k=1}^{n} k^2 = (1/3 + o(1))n^3$ and $\sum_{k=1}^{n} k^4 = (1/5 + o(1))n^5$, we have

$$\log f_n = -\left(\frac{1}{6} + o(1)\right) n^3 n^{-(3-2\varepsilon)} + \left(\frac{1}{120} + o(1)\right) n^5 n^{-(6-4\varepsilon)}.$$

Letting $c = 1/6$, $f_n \le \exp(-c(1 + o(1))n^{2\varepsilon} + c(1 + o(1))n^{-1+4\varepsilon})$, and recalling $\varepsilon < 1/4$, $f_n \le \exp(-c(1 + o(1))n^{2\varepsilon})$ as desired.

Case 2. $\frac{1}{n} \le |t| \le \frac{\pi}{n}$*:*

Here we use will the monotonicity of $f_n(t)$ in n. It follows directly from $f_n(t) = \prod_{k=1}^{n} \cos(kt)$ and $0 \le \cos(x) \le 1$ that $|f_n(t)| \le |f_m(t)|$ for $n \ge m$. Let $h_n = \lfloor n/4 \rfloor$ be the greatest integer in $n/4$. Then $|f_n(t)| \le |f_{h_n}(t)|$. From Case 1, $f_{h_n}(t) \le \exp(-ch_n^{2\varepsilon}(1 + o(1)))$ for $h_n^{-(\frac{3}{2}-\varepsilon)} \le |t| \le 1/h_n$. Since $1/n > h_n^{-5/4} \ge h_n^{-(\frac{3}{2}-\varepsilon)}$ for $n > 1050$ and $h_n \le n/4 \le n/\pi$ implies $\pi/n \le 1/h_n$, we get $|f_n(t)| \le \exp(-ch_n^{2\varepsilon}(1 + o(1)))$ for $t \le \pi/n$ as $n \to \infty$.

Case 3. $\frac{\pi}{n} \leq |t| \leq \frac{\pi}{2}$*:*

Note that it suffices to show that $|f_n(t)| \leq c^n$ for a constant $c < 1$, since then

$$\int_{\pi/n \leq |t| < \pi/2} |f_n(t)|\, dt \leq \pi \cdot c^n = o(n^{-3/2}).$$

To accomplish this, we first transform $f_n(t)$ from a product to a sum using the arithmetic-geometric mean inequality:

$$(f_n^2(t))^{1/n} = \left(\prod_{k=1}^{n} \cos^2(kt) \right)^{1/n} \leq \frac{1}{n} \sum_{k=1}^{n} \cos^2(kt). \tag{1}$$

The sum on the right-hand side can be simplified as

$$\sum_{k=1}^{n} \cos^2(kt) = \frac{n}{2} + \frac{1}{2} \sum_{k=1}^{n} \cos(2kt) = \frac{n}{2} + \frac{\cos((n+1)t)}{2} \frac{\sin(nt)}{\sin(t)}. \tag{2}$$

Combining equations 1 and 2, we can write

$$|f_n(t)| \leq \left(\frac{1}{2} + \frac{1}{2n} \frac{1}{\sin(t)} \right)^{n/2}. \tag{3}$$

We will now apply the Jordan-style concavity inequality $|\sin(t)| \geq \frac{2|t|}{\pi}$ for $0 \leq |t| \leq \pi/2$. For $\pi/n \leq |t| \leq \pi/2$, substitution in equation 3 gives:

$$|f_n(t)| \leq \left(\frac{1}{2} + \frac{1}{2n} \frac{\pi}{2|t|} \right)^{n/2} = \left(\frac{1}{2} + \frac{\pi}{4n|t|} \right)^{n/2}.$$

Observing that the right-hand side is monotonically decreasing in $|t|$, we have $|f_n(t)| \leq f_n(\pi/n)$. Evaluating, we see

$$|f_n(t)| \leq \left(\frac{1}{2} - \frac{1}{2n} \right)^{n/2}$$

proving $|f_n(t)| \leq (\sqrt{7/16})^n$ (since we may assume $2n \geq 16$ as $n \to \infty$).

□

We will also need the following straightforward lemma from analysis.

Lemma 3. *Let $c \in \mathbb{R}$ and $a(c), b(c)$ be real-valued functions such that*

$$\lim_{c \to \infty} -a(c)\sqrt{c} = \lim_{c \to \infty} b(c)\sqrt{c} = \infty.$$

Then

$$\int_{a(c)}^{b(c)} e^{-ct^2}\, dt \sim \int_{-\infty}^{\infty} e^{-ct^2}\, dt$$

as $c \to \infty$.

3 Main Result

We now prove Conjecture 1 holds.

Theorem 4. *When $n \equiv 0$ or $3 \pmod 4$, $S(n) \sim \sqrt{6/\pi} \cdot 2^n n^{-3/2}$.*

Proof. When $n \equiv 0$ or $3 \pmod 4$, $f_n(t + m\pi) = f_n(t)$ for any integer m, so

$$S(n) = \frac{2 \cdot 2^{n-1}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_n(t)\, dt, \tag{4}$$

and we may assume $|t| \leq \pi/2$ when evaluating $f_n(t)$.

By Lemma 2, $\int_{n^{-(3/2-\varepsilon)}<|t|<\pi/2} |f_n(t)|\, dt = o(n^{-3/2})$, so it suffices to consider $|t| < n^{-(3/2-\varepsilon)}$ when estimating $f_n(t)$ around $t = 0$. Recalling

$$f_n(t) = \prod_{k=1}^{n} e^{\ln(\cos(kt))},$$

we first use Taylor series to approximate $g_k(t) = \ln(\cos(kt))$ at $t = 0$. We have $g_k(t) = -k^2t^2/2 + R_2$, where R_2 is the Lagrange remainder. Then R_2 is bounded by a constant times $t^3 g_k^{(3)}(t_0)$ for some t_0 near 0. Since $g_k^{(3)}(t) = -2k^3 \sin(kt)/\cos^3(kt)$, and t_0 is small (since $|t| < n^{-(3/2-\varepsilon)}$), we have that $R_2 \leq ak^3t^3$ where a is constant. The absolute error for $g_k(t)$ is thus bounded by $ak^3n^{-(9/2-3\varepsilon)}$.

Around $t = 0$, $f_n(t)$ can be approximated as $\delta \prod_{k=1}^{n} e^{-\frac{k^2t^2}{2}}$ with error $\delta \leq \prod_{k=1}^{n} e^{ak^3n^{-(9/2-3\varepsilon)}}$. This simplifies to

$$f_n(t) \approx e^{-t^2/2 \sum_{k=1}^{n} k^2} = e^{-t^2 n(n+1)(2n+1)/12}. \tag{5}$$

Our error bound simultaneously simplifies to

$$\delta \leq e^{an^{-(9/2-3\varepsilon)} \sum_{k=1}^{n} k^3} = e^{an^{-(9/2-3\varepsilon)} n^2(n+1)^2/4}.$$

This proves that the error goes to one as n approaches infinity whenever $\varepsilon < \frac{1}{6}$.

Substituting (5) for $f_n(t)$ in equation 4, and applying Lemma 2, we find that

$$\frac{\pi S(n)}{2^n} = (1 + o(1)) \int_{-n^{-(3/2-\varepsilon)}}^{n^{-(3/2-\varepsilon)}} e^{-n(n+1)(2n+1)t^2/12}\, dt + o(n^{-3/2}).$$

By Lemma 3, this implies

$$\frac{\pi S(n)}{2^n} = (1 + o(1)) \int_{-\infty}^{\infty} e^{-n(n+1)(2n+1)t^2/12}\, dt + o(n^{-3/2}).$$

Using

$$\int_{-\infty}^{\infty} e^{-Ct^2}\, dt = \sqrt{\frac{\pi}{C}}$$

for any constant $C > 0$ and $n(n+1)(2n+1) \sim 2n^3$, we have

$$S(n) \sim \frac{2^n}{\pi} \sqrt{\frac{12\pi}{2n^3}} = \sqrt{6/\pi} \cdot 2^n n^{-3/2},$$

as desired.

$\square$

4 Acknowledgements

This research was partially completed while the author was an intern at Microsoft Research, Redmond, WA. We thank Henry Cohn for suggesting the problem and providing comments which greatly improved the manuscript. Special thanks are due to Xavier Gourdon, who pointed out a flaw in an earlier version of the paper, and suggested the arithmetic-geometric mean inequality approach now used in the third case of Lemma 2.

References

[1] Dorin Andrica and Ioan Tomescu, "On an Integer Sequence Related to a Product of Trigonometric Functions, and its Combinatorial Relevance," *Journal of Integer Sequences*, vol. 5 (2002), Article 02.2.4.

[2] N.G. de Bruijn, Asymptotic Methods in Analysis, Dover Publications, Inc., New York, 1981.

(Concerned with sequence A025591.)